ON BODIES OF MINIMUM WAVE DRAG IN NONUNIFORM FREE STREAM OF GAS

(O TELAKH S MINIMAL'NYM VOLNOVYM SOPROTIVLENIEM V NERAVNOMERNOM NABEGAIUSHCHEM POTOKE GAZA)

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We shall consider the problem of constructing the generatrix of a body of revolution ab (Fig.1) with minimum wave drag in nonuniform axisymmetric flow. We assume that the free stream as well as coordinates of the points a and b are given, and moreover, that an attached shock wave ac occurs. Let bc be a characteristic of the second family, and cd of the first family. The problem is to be solved under the assumption that in the triangle abc the flow is supersonic and there is no shock wave.

A similar problem with an attached shock wave has been studied in [1] for the case of uniform free stream. For a nonuniform free stream, the problem posed has been considered in [2], but an error has been made there. In counting the number of conditions and the arbitrary quantities of the problem, conditions on the shock wave at the point $\psi = \psi_c$ has not been accounted for. Instead, the trnsversality condition was used, which is identically satisfied at that point because of the extremal conditions and the correlations on the shock wave. Consequently, the number of conditions and the arbitrary quantities coincide, and the incorrect conclusion was drawn that the problem is solvable.

The gas flow is described by Equations

$$\frac{\partial r\rho w\cos\vartheta}{\partial x}+\frac{\partial r\rho w\sin\vartheta}{\partial r}=0,\qquad \frac{\partial}{\partial x}r\,(p+\rho w^2\cos^2\vartheta)+\frac{\partial}{\partial r}r\rho w^2\sin\vartheta\cos\vartheta=0 \tag{1}$$

$$\frac{w^2}{2}+\frac{\varkappa}{\varkappa-1}\frac{p}{\rho}=\frac{1}{2}\frac{\varkappa+1}{\varkappa-1},\qquad \frac{p}{\rho^\varkappa}=\varphi^{\varkappa-1}(\psi) \tag{2}$$

Here x and r are the Cartesian coordinates in the meridional plane of flow, ϑ is the angle between the velocity vector and the x-axis, $\varkappa$ is the adiabatic coefficient, w is the absolute magnitude of the velocity, ρ is the gas density, p is the pressure, and ψ is the stream function; and

$$d\psi = r\rho w\,(\cos\vartheta\,dr-\sin\vartheta\,dx)$$

The free stream is given in the functions $w_0(x,r)$, $\rho_0(x,r)$ and $\vartheta_0(x,r)$. In what follows, we shall take the functions before the shock wave as that along the shock wave. The coordinates of the shock wave ac will be denoted by x° and r°, while those of the characteristic bc by x and r.

Let χ be the wave drag of the body of revolution with generatrix ab, divided by 2π. The quantity χ is expressed by contour integrals along ac and bc with the help of the second equation of (1)

$$\chi = \int_{\psi_a}^{\psi_c} \left\{ \frac{\varkappa+1}{2\varkappa}\left(w_0 + \frac{1}{w_0}\right)\frac{\sin\sigma}{\sin(\sigma-\vartheta_0)} - \right.$$

$$\left. - w_0 \sin\vartheta_0 \cot(\sigma-\vartheta_0) - a(\alpha)\left[\cos\vartheta - \frac{1}{\varkappa}\sin\alpha\sin(\vartheta-\alpha)\right]\right\} d\psi \qquad (3)$$

The distance between the points a and b along the x-axis is also expressed by contour integrals along ac and bc

$$X = \int_{\psi_a}^{\psi_c} \left[\frac{\cos\sigma}{r^\circ \rho_0 w_0 \sin(\sigma-\vartheta_0)} + \frac{\varphi(\sigma, x^\circ, r^\circ)}{\sqrt{\varkappa r}}\,\tau(\alpha)\cos(\vartheta-\alpha)\right] d\psi \qquad (4)$$

In Formulas (3) and (4)

$$a(\alpha) = \left(\frac{\varkappa+1}{\varkappa-\cos 2\alpha}\right)^{\frac{1}{2}}, \qquad \tau(\alpha) = \left(\frac{\varkappa+1}{2\varkappa}\frac{1-\cos 2\alpha}{\varkappa-\cos 2\alpha}\right)^{-\frac{1}{2}\frac{\varkappa+1}{\varkappa-1}}, \qquad \rho w^2 \sin^2\alpha = \varkappa p$$

Here α is the Mach angle, and σ is the angle of inclination of the shock wave to the x-axis.

The function $\varphi(\sigma, x^\circ, r^\circ)$ is determined from (2) by the correlations on the shock wave involving the inclination of the shock and the free flow. Moreover, on the characteristic bc, we have

$$\frac{dr}{d\psi} + \frac{\varphi(\sigma, x^\circ, r^\circ)}{\sqrt{\varkappa r}}\,\tau(\alpha)\sin(\vartheta-\alpha) = 0 \qquad (5)$$

$$\frac{d\vartheta}{d\psi} - \frac{1+\cos 2\alpha}{\varkappa-\cos 2\alpha}\frac{d\alpha}{d\psi} - \frac{\sin\vartheta\sin\alpha}{r\sin(\vartheta-\alpha)}\frac{dr}{d\psi} + \frac{\sin 2\alpha}{2\varkappa}\frac{d\ln\varphi}{d\psi} = 0 \qquad (6)$$

The position of the shock wave is defined by Equations

$$\frac{dr^\circ}{d\psi} - \frac{\sin\sigma}{r^\circ\rho_0 w_0 \sin(\sigma-\vartheta_0)} = 0, \qquad \frac{dx^\circ}{d\psi} - \frac{\cos\sigma}{r^\circ\rho_0 w_0\sin(\sigma-\vartheta_0)} = 0 \qquad (7)$$

We consider the following variational problem: for given r_a, r_b and X, and for a given free flow, to find the functions $\alpha(\psi)$, $\vartheta(\psi)$, $\sigma(\psi)$, $r(\psi)$, $x^\circ(\psi)$, and $r^\circ(\psi)$, which render Expression (3) an extremum, while satisfying the isoperimetric condition (4) and the differential equations (5) to (7). Moreover, at the point $\psi = \psi_c$, conditions

$$\alpha(\psi_c) = \alpha(\sigma_c, x_c^\circ, r_c^\circ), \qquad \vartheta(\psi_c) = \vartheta(\sigma_c, x_c^\circ, r_c^\circ)$$

obtained from the shock wave correlations must be satisfied.

As free functions we choose α and σ, and the remaining are connected with these by the differential relations.

Using the method of lagrange multipliers, the problem reduces to one of finding the extremum of some functional without conditions. From the vanishing of the first variation of this functional, we obtain the equations which must be satisfied by the functions being sought

$$\lambda(\varkappa\sin 2\vartheta + \sin 2\alpha) + \varkappa\lambda_3(1-\cos 2\vartheta) = 0 \qquad (8)$$

$$\lambda\varphi(\sigma, x^\circ, r^\circ)\,\tau(\alpha)\cos\alpha - \sqrt{\varkappa}\,a(\alpha)\,r\sin^2\vartheta = 0$$

$$\frac{(\varkappa+1)}{2\varkappa}\frac{(W_0^2-1)\sin\vartheta_0}{w_0\sin^2(\sigma-\vartheta_0)} + g\frac{\lambda_1\sin\vartheta_0 - (\lambda-\lambda_2)\cos\vartheta_0}{\sin(\sigma-\vartheta_0)} + G(\alpha,\vartheta)\,\varphi_\sigma' = 0$$

$$\frac{dx^\circ}{d\psi} - g\cos\sigma = 0, \qquad \frac{dr^\circ}{d\psi} - g\sin\sigma = 0$$

$$\frac{dr}{d\psi}+\frac{\varphi(\sigma,x^\circ,r^\circ)\,\tau(\alpha)\sin(\vartheta-\alpha)}{\sqrt{\varkappa r}}=0,\qquad \frac{d\lambda_3}{d\psi}+\frac{G(\alpha,\vartheta)}{r}=0 \tag{8 cont.}$$

$$\frac{d\lambda_2}{d\psi}-\chi_1\frac{\partial\vartheta_0}{\partial x^\circ}-\chi_2\frac{\partial w_0}{\partial x^\circ}+\chi_3\frac{\partial\rho_0}{\partial x^\circ}=0,\quad \frac{d\lambda_1}{d\psi}-\chi_1\frac{\partial\vartheta_0}{\partial r^\circ}-\chi_2\frac{\partial w_0}{\partial r^\circ}+\chi_3\frac{\partial\rho_0}{\partial r^\circ}+\frac{F}{r^\circ}=0$$

Here λ is the constant multiplier, while $\lambda_1(\psi), \lambda_2(\psi), \lambda_3(\psi)$ are the variable Lagrange multipliers

$$\varphi_\sigma'=\frac{1}{\varphi}\frac{\partial\varphi}{\partial\sigma},\quad g=\frac{1}{r^\circ\rho_0 w_0\sin(\sigma-\vartheta_0)},\quad F=g\left[(\lambda-\lambda_2)\cos\sigma-\lambda_1\sin\sigma\right]$$

$$W_0^2=\frac{\varkappa-1}{\varkappa+1}w_0^2,\quad G(\alpha,\vartheta)=\frac{a(\alpha)}{\varkappa}\tan\alpha\left[\varkappa\sin\vartheta-\cos\alpha\sin(\vartheta-\alpha)\right]$$

$$\chi_1=\frac{\varkappa+1}{2\varkappa}\frac{\sin\sigma\cos(\sigma-\vartheta_0)}{\sin^2(\sigma-\vartheta_0)}\left(w_0+\frac{1}{w_0}\right)-\frac{\sin\vartheta_0}{\sin^2(\sigma-\vartheta_0)}w_0+$$

$$+(F-w_0\cos\vartheta_0)\cot(\sigma-\vartheta_0)-G(\alpha,\vartheta)\varphi_\sigma'$$

$$\chi_2=G(\alpha,\vartheta)\frac{2w_0}{\varkappa-1}\left\{\frac{4\varkappa\sin^2(\sigma-\vartheta_0)+(\varkappa-1)^2}{4w_0^2\sin^2(\sigma-\vartheta_0)-(\varkappa^2-1)(1-W_0^2)}-\right.$$

$$\left.-\frac{\varkappa}{w_0^2\left[1-W_0^2\cos^2(\sigma-\vartheta_0)\right]}\right\}+\frac{\varkappa\sin(\sigma-2\vartheta_0)+\sin\sigma}{2\varkappa\sin(\sigma-\vartheta_0)}-$$

$$-\frac{F}{w_0}-\frac{(\varkappa+1)\sin\sigma}{2\varkappa w_0^2\sin(\sigma-\vartheta_0)},\qquad \chi_3=\frac{F+G(\alpha,\vartheta)}{\rho_0}$$

The Lagrange multiplier, corresponding to relation (6), is identical to zero [1 and 3].

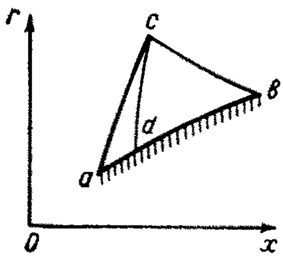

Fig. 1

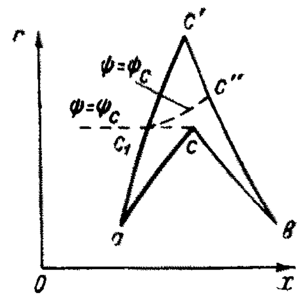

Fig. 2

Thus, we have obtained a system of equations (8) for the nine unknowns: $\alpha, \vartheta, \sigma, x^\circ, r^\circ, r, \lambda_1, \lambda_2$ and λ_3 ; as boundary conditions, we have

$$\alpha(\psi_c)=\alpha(\sigma_c, x_c^\circ, r_c^\circ),\qquad \vartheta(\psi_c)=\vartheta(\sigma_c, x_c^\circ, r_c^\circ) \tag{9}$$

$$r(\psi_c)=r^\circ(\psi_c),\qquad x^\circ(\psi_a)=x_a,\qquad r^\circ(\psi_a)=r_a,\qquad r(\psi_a)=r_b \tag{10}$$

$$\lambda_2(\psi_c)=0,\qquad \lambda_1(\psi_c)=-\lambda_3(\psi_c) \tag{11}$$

In addition, the functions being sought for must satisfy the isoperimetric condition (4) and the transversality condition at $\psi=\psi_c$

$$\frac{\varkappa+1}{2\varkappa}\left(w_0+\frac{1}{w_0}\right)\frac{\sin\sigma}{\sin(\sigma-\vartheta_0)}-w_0\sin\vartheta_0\cot(\sigma-\vartheta_0)-$$
$$-a(\alpha)\left[\cos\vartheta-\frac{1}{\varkappa}\sin\alpha\sin(\vartheta-\alpha)\right]+ \qquad (12)$$
$$+\lambda\left[g\cos\sigma+\frac{\varphi(\sigma,x^\circ,r^\circ)}{\sqrt{\varkappa r}}\tau(\alpha)\cos(\vartheta-\alpha)\right]+$$
$$+\lambda_3\left[g\sin\sigma+\frac{\varphi(\sigma,x^\circ,r^\circ)}{\sqrt{\varkappa r}}\tau(\alpha)\sin(\vartheta-\alpha)\right]=0$$

Here conditions (9) are the correlations on the shock wave at the point ψ_c . Conditions (10) follow from examination of Fig.1. Conditions (11) are taken, by virtue of the arbitrary choice of the Lagrange multipliers, such that the first variation of the functional obtained vanish. In obtaining conditions (11) and (12), we used the relationship between the variations δr at $\psi=\psi_c$ and $\delta\psi_c$, obtained in the following manner. From Fig.2, it is obvious that

$$\delta r|_{\psi=\psi_c}=r_{c''}-r_c$$

Assume that for a variation of position, the point c moves to c'. Then the stream line at $\psi=\psi_c$ intersects the varied shock wave at c_1 and the characteristic at c''. Letting

$$r_{c_1}{}^\circ-r_c=\delta r^\circ|_{\psi=\psi_c}$$

we have

$$r_{c_1}{}^\circ=r_{c'}{}^\circ-\frac{dr^\circ}{d\psi}\bigg|_{ac}\delta\psi_c,\qquad r_{c''}=r_{c'}-\frac{dr}{d\psi}\bigg|_{bc}\delta\psi_c$$

At the point c', $r^\circ=r$. Hence,

$$\delta r|_{\psi=\psi_c}=\left(\frac{dr^\circ}{d\psi}\bigg|_{ac}-\frac{dr}{d\psi}\bigg|_{bc}\right)\delta\psi_c+\delta r^\circ|_{\psi=\psi_c}$$

Here the double indices ac and bc indicate respectively the derivatives of r° along the shock wave ac at the point c , and the derivative r along the characteristic (of the second family) bc at the point c

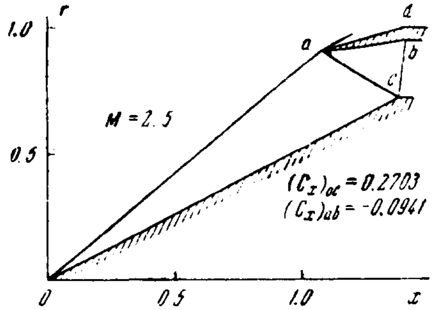

Fig. 3

It should be pointed out that the transversality condition (12) is identically satisfied by virtue of the shock wave correlations and the first two equations of system (8). Thus the number of conditions drops by one. There are eight arbitrary quantities in the functions being determined: six of them occur from the six differential equations of (8), and in addition there are two arbitrary quantities λ and ψ_c. The number of conditions (9), (10), (11) and (4) equals nine. Consequently, the variational problem as posed has no solution if a two-sided extremum is being sought for. However, for some particular relations between the quantities w_0, ϑ_0, ρ_0, which characterize the free flow, and also r_a/X and r_b/X , the problem does admit a solution. This, obviously, occurs when the last equations of system (8) are satisfied at the point ψ_c because of the shock wave correlations. Excluding from these equations λ and λ_3, we obtain

$$\frac{(\varkappa+1)(W_0^2-1)\sin\vartheta_0}{w_0\sin^2(\sigma-\vartheta_0)} + a(\alpha)\tan\alpha\left\{2\varphi_\sigma'\,[\varkappa\sin\vartheta-\cos\alpha\sin(\vartheta-\alpha)] + \frac{2\varkappa\sin\vartheta\sin(\vartheta_0-\vartheta)+\sin 2\alpha\sin\vartheta_0}{\sin(\sigma-\vartheta)\sin(\sigma-\vartheta_0)}\right\}=0 \qquad (13)$$

The solution of the problem exists when Equation (13) is satisfied at the point $\psi = \psi_c$ by virtue of the shock wave correlations. This means that if the free flow is given in the functions w_0, ϑ_0, ρ_0, then Equation (13) determines the value of σ at the point ψ_c, such that the problem possesses a solution. Let some nonuniform flow be given, and let us consider Equation (13) at some point in this flow. The equation has discrete roots. From them we must select such values of σ, which satisfies the conditions of the problem; in other words, the inclination of the shock must be greater than that of the Mach line and smaller than that value of σ for which the velocity behind the shock becomes sonic. If at least one such root is found, then we may draw the extremal shock and characteristic from this point by integrating the system (8). Indeed, if at the point ψ_c the value σ is known, then we can determine, at that point α and ϑ from the shock wave correlations, and then λ_1, λ_2 and λ_3 from the last equation of (8). We thus have a Cauchy problem. The integration must be carried to a value of ψ which is stipulated by the conditions of each concrete problem. In addition, σ must not exceed the value σ^* at which the velocity behind the shock becomes sonic. Should this occur, then the integration can only be carried to the value of ψ at which $\sigma = \sigma^*$.

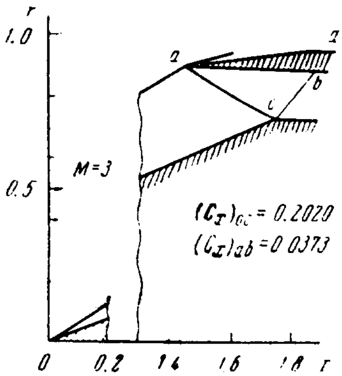

Fig. 4

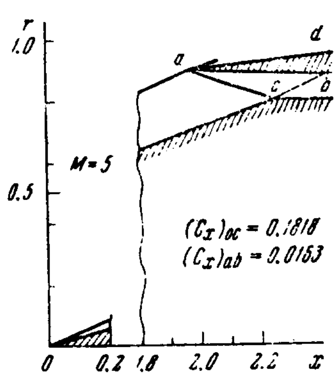

Fig. 5

Having constructed the shock wave ac and the characteristic bc satisfying the Euler equations (8), the flow calculations may then be carried out by the method of characteristics. We first solve the Cauchy problem for the gas dynamic equations (1) and (2) with data on the shock wave ac, which permits the determination of the characteristic cd; then we solve the Goursat problem with data on the characteristics cd and bc. All the streamlines (ψ = const) are the desired profiles.

In Fig. 3, 4 and 5 are shown examples of analysis of the inner wall of the lip of supersonic center-body diffusers, having extremal wave drag. In the above examples a cone is taken as the center-body. In the figures, the following notation is used: oc is the cone generatrix, oa is the conical shock, ac is the shock corresponding to the extremum condition, bc is the characteristic of the first family, and ab is the profile being sought for. In the examples the generatrix of the outer wall of the diffusor lip ad is taken to be an arbitrary straight line. Also exhibited are the values of c_x (C_D – drag coefficient) for the cone and for the inner wall. The quantity c_x refers to the area πr_a^2. In the cases of $M = 3$ and $M = 5$, the angle ϑ is zero on the characteristic bc

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BIBLIOGRAPHY

1. Shmyglevskii, Iu.D., Ob odnom klasse tel vrashcheniia s minimal'nym volnovym soprotivleniem (On a class of bodies of revolution with minimum wave drag). *PMM* Vol.24, № 5, 1960.
2. Kostychev, G.I., K resheniiu odnoi variatsionnoi zadachi sverkhzvukovykh techenii (On the solution of a variational problem of supersonic flow). Izv.vyssh.uchebn.zaved., MVO, ser.aviats.tekhn., № 3, 1958
3. Shmyglevskii, Iu.D., Nekotorye variatsionnye zadachi gazovoi dinamiki osesimmetrichnykh sverkhzvukovykh techenii (Some variational problems in the gas dynamics of axisymmetric supersonic flows). *PMM* Vol.21, № 2, 1957.

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